GROUP THEORY 2024 - 25, SOLUTION SHEET 7

Exercise 1. Review the lecture and understand/fill in the gaps in the proofs.

Exercise 2. By the correspondence theorem, normal subgroups of G that contain H are in bijection with normal subgroups of G/H. This proves both implications of the claim.

Exercise 3. A composition series is given by

$$0 = 12\mathbb{Z}/12\mathbb{Z} \trianglelefteq 6\mathbb{Z}/12\mathbb{Z} \trianglelefteq 3\mathbb{Z}/12\mathbb{Z} \trianglelefteq \mathbb{Z}/12\mathbb{Z}$$

with composition factors

$$\{\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}\}.$$

The composition series is not unique, for example here is another one

$$0 = 12\mathbb{Z}/12\mathbb{Z} \trianglelefteq 6\mathbb{Z}/12\mathbb{Z} \trianglelefteq 2\mathbb{Z}/12\mathbb{Z} \trianglelefteq \mathbb{Z}/12\mathbb{Z}$$

They have the same composition factors by a theorem of the lectures.

Exercise 4. Let $V_4 = \{1, (12)(34), (13)(24), (14)(23)\}$ be the Klein four-group. Notice that it is precisely the subgroup of A_4 elements of order 2. Since for all $\sigma \in A_4$ and $x \in V_4$ we have $(\sigma x \sigma^{-1})^2 = 1$, this prove that $\sigma x \sigma^{-1}$ has order 2 and hence belong to V_4. This shows that V_4 is normal in A_4. It follows that

$$0 \trianglelefteq \mathbb{Z}/2\mathbb{Z} = \langle (12)(34) \rangle \trianglelefteq V_4 \trianglelefteq A_4$$

is a composition series. Its composition factors are

$$\{\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}\}$$

since those are the only groups with the required cardinalities. Since $A_4 \trianglelefteq S_4$ is normal, we can extend it to a composition series

$$0 \trianglelefteq \mathbb{Z}/2\mathbb{Z} = \langle (12)(34) \rangle \trianglelefteq V_4 \trianglelefteq A_4 \trianglelefteq S_4$$

with composition factors

$$\{\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}\}.$$

Exercise 5. By the properties of semi-direct products, we have an exact sequence:

$$1 \to G \to G \rtimes_\varphi H \to H \to 1.$$

Then it follows from Proposition 22 of the notes that the composition factors of $G \rtimes_\varphi H$ are just the compositions factors of G and the composition factors of H.

Exercise 6. (1) By exercise 5 of last week, we know that we can write

$$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \mathbb{Z}/p_2^{a_2}\mathbb{Z} \times \ldots \times \mathbb{Z}/p_k^{a_k}\mathbb{Z}.$$

Hence a composition series is given by the following:

$$0 \trianglelefteq \mathbb{Z}/p_1\mathbb{Z} \trianglelefteq \mathbb{Z}/p_1^2\mathbb{Z} \trianglelefteq \mathbb{Z}/p_1^3\mathbb{Z} \trianglelefteq \ldots \trianglelefteq \mathbb{Z}/p_1^{a_1}\mathbb{Z} \trianglelefteq \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \mathbb{Z}/p_2\mathbb{Z} \trianglelefteq$$
$$\mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \mathbb{Z}/p_2^2\mathbb{Z} \trianglelefteq \ldots \trianglelefteq \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \mathbb{Z}/p_2^{a_2}\mathbb{Z} \times \ldots \times \mathbb{Z}/p_k^{a_k}\mathbb{Z}$$

which has length $a_1 + a_2 + \ldots + a_k$. The composition factors consist of a_i-times $\mathbb{Z}/p_i\mathbb{Z}$ for all $1 \leq i \leq k$.

(2) Let $n \in \mathbb{N}$. Using proposition 19 of the lectures, we know that $G = \mathbb{Z}/n\mathbb{Z}$ has a composition series

$$0 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_k = G.$$

Since G is abelian, so are his subgroups. Hence the composition factors G_{i+1}/G_i are finite simple abelian groups, i.e. they are cyclic of prime order (as explained in the lectures). It follows that

$$n = |G| \cong |G/G_{k-1}| \times |G_{k-1}| \cong |G/G_{k-1}| \times |G_{k-1}/G_{k-2}| \times |G_{k-2}|$$
$$\cong \prod_{i=0}^{k-1} |G_{i+1}/G_i|$$

which is a product of primes. By the Jordan Hölder theorem, the composition factors G_{i+1}/G_i are unique (up to permuting the factors), which shows that such a decomposition of n as a product of primes is unique.

Exercise 7. By exercise 7 of sheet 4 we have an isomorphism $D_{2n} \cong \mathbb{Z}/n\mathbb{Z} \times_\varphi \mathbb{Z}/2\mathbb{Z}$. Hence we have a short exact sequence

$$0 \to \mathbb{Z}/n\mathbb{Z} \to D_{2n} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

and so by the lectures we know that D_{2n} has a composition series given by attaching a composition series of $\mathbb{Z}/n\mathbb{Z}$ with one of $\mathbb{Z}/2\mathbb{Z}$. The previous exercise gives us such composition series. Moreover, exercise 5 tells us that the composition factors is the union of the factors of those two groups.

Exercise 8. Suppose by contradiction that we have the existence of a proper normal subgroup $H \trianglelefteq G$. Then if we let $G_0 := 1$, there exists $n \in \mathbb{N}$ such that $G_n \subseteq H$ and $G_{n+1} \nsubseteq H$. However, we then have that $G_{n+1} \bigcap H$ is a proper normal subgroup of G_{n+1}, which contradicts the assumption. Consider the inclusions $A_5 \subset A_6 \subset \ldots$ of the alternating groups, all of which are simple. Then

$$\bar{A} = \bigcup_{i=5}^{\infty} A_i$$

is infinite and simple.

Exercise 9. (1) See the proof of propositions $20, 21, 22$ and the Jordan-Holder theorem.

(2) We have a short exact sequence

$$1 \to K \to G \to G/K \to 1$$

By the previous point, $\text{length}(G) = \text{length}(K) + \text{length}(G/K)$, but as K is a proper subgroup of G, we have that G/K is not trivial and thus of length strictly greater than 0. This implies that $\text{length}(G) > \text{length}(K)$.

(3) If we have a strict chain

$$1 \trianglelefteq G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \dots$$

made up of normal subgroups of G, we can apply (2) to get

$$0 < \text{length}(G_0) < \text{length}(G_1) < \dots$$

Thus, any such chain must be finite and have length at most equal to $\text{length}(G) + 1$.

(4) Let us prove each of the two implications:

" $\implies$ " Observe that the reasoning of (3) still holds, even if the chains described in (a) and (b) are not made up of normal subgroups of G.

" $\impliedby$ " Clearly, G is normal in G. If G is simple, we are already done. If not, pick a normal subgroup $H_0 \trianglelefteq G$. If H_0 is maximal in G, we stop. If not, we continue iterating this process by choosing at each time a normal subgroup H_i of G such that $H_{i-1} \trianglelefteq H_i$ with strict inclusion. By assumption (b), this process must terminate at some H_n for n a positive integer. Set $G_1 = H_n \trianglelefteq G$. One can check that G_1 is maximal in G, if not the above process would have not terminated. Observe that G_1 is also normal in G, by construction, Thus, we can inductively apply the same reasoning as above to obtain a descending normal chain in which each inclusion is maximal:

$$G \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \dots$$

By assumption (a) such a chain must stabilize at some G_n and by construction of the G_i we must have that $G_n = 1$. We have thus obtained a composition series for G, so G has finite length.